ELSEVIER
Discrete Applied Mathematics 95 (1999) 311–320

DISCRETE
APPLIED
MATHEMATICS

Asymptotic results on suborthogonal $\vec{\mathfrak{G}}$-decompositions of complete digraphs

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Received 1 November 1997; accepted 23 December 1998

Abstract

A $\vec{\mathfrak{G}}$-decomposition of a complete digraph $\vec{\mathfrak{D}}_n$ is a partition of $\vec{\mathfrak{D}}_n$ into isomorphic copies (called pages) of $\vec{\mathfrak{G}}$. A $\vec{\mathfrak{G}}$-decomposition is said to be suborthogonal if the union of any two distinct pages contains at most one pair of reverse arcs. Wilson (Proceedings of the fifth British Combinatorial Conference, 1975, pp. 647–659) proved in 1975 that a $\vec{\mathfrak{G}}$-decomposition exists for almost all integers n satisfying certain necessary conditions. In this paper we shall prove that under the same conditions there exists even a suborthogonal $\vec{\mathfrak{G}}$-decomposition.

Keywords: Suborthogonal decomposition; Edge partition; Complete digraph

1. Introduction

For any positive integer n, let $\vec{\mathfrak{D}}_n$ denote the complete digraph on n vertices. A *decomposition* of $\vec{\mathfrak{D}}_n$ is a family $\mathscr{G}=\{\vec{\mathfrak{G}}_1,\dots,\vec{\mathfrak{G}}_s\}$ of subdigraphs of $\vec{\mathfrak{D}}_n$ (called *pages*), such that each arc of $\vec{\mathfrak{D}}_n$ belongs to exactly one page of $\mathscr{G}$. If, in addition, all pages are isomorphic to a given digraph $\vec{\mathfrak{G}}$, we call $\mathscr{G}$ a $\vec{\mathfrak{G}}$-*decomposition* of $\vec{\mathfrak{D}}_n$.

$\vec{\mathfrak{G}}$-decompositions have been considered widely in literature, see e.g. [1] and its references. Necessary conditions for the existence of a $\vec{\mathfrak{G}}$-decomposition are well known. For any vertex w in $\vec{\mathfrak{G}}=(W,B)$, let $deg^-(w)$ and $deg^+(w)$ denote the indegree and the outdegree of w in $\vec{\mathfrak{G}}$, respectively. Further, let d be the solution of the integer linear program

$$d=\min Y=\min\Big\{y\in\mathbb{Z}:\ y\geqslant 1 \text{ and } \forall w\in W\ \exists x_w\in\mathbb{Z} \text{ such that}$$

$$\sum_{w\in W} deg^-(w)x_w=\sum_{w\in W} deg^+(w)x_w=y\Big\},$$

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0166-218X/99/$ - see front matter
PII: S0166-218X(99)00083-9

or $d=0$ if $\vec{\mathfrak{G}}$ has no arcs. It is noteworthy, that d is just the greatest common divisor of the integers in Y, if $d>0$.

Necessary conditions for the existence of a $\vec{\mathfrak{G}}$-decomposition of $\vec{\mathfrak{D}}_n$ are

$$n-1 \equiv 0 \bmod d, \tag{1}$$

$$n(n-1) \equiv 0 \bmod b, \tag{2}$$

where b denotes the number of arcs in $\vec{\mathfrak{G}}$. Also, by a result of Wilson [5], these conditions are known to be sufficient whenever n is large enough.

Gronau et al., introduced in [2] the concept of orthogonal $\vec{\mathfrak{G}}$-decompositions (or orthogonal $\vec{\mathfrak{G}}$-covers). A decomposition $\mathscr{G}$ of $\vec{\mathfrak{D}}_n$ is said to be *orthogonal* if the union of any two distinct pages contains precisely one digon, i.e. a digraph isomorphic to $\vec{\mathfrak{D}}_2$. This precondition happens to be rather restrictive: Every non-trivial orthogonal $\vec{\mathfrak{G}}$-decomposition of $\vec{\mathfrak{D}}_n$ consists of n pages, and $\vec{\mathfrak{G}}$ has exactly $n-1$ arcs.

Our objective is to study the larger class of suborthogonal $\vec{\mathfrak{G}}$-decompositions. We call a decomposition $\mathscr{G}$ of $\vec{\mathfrak{D}}_n$ *suborthogonal* if the union of any two distinct pages contains at most one digon.

Obviously, for every digraph $\vec{\mathfrak{G}}$ there exists a suborthogonal $\vec{\mathfrak{G}}$-decomposition $\mathscr{G}$ of the complete digraph $\vec{\mathfrak{D}}_1$. (Just choose the empty family $\mathscr{G}$.) On the other hand, if $\vec{\mathfrak{G}}$ has no arcs, $\vec{\mathfrak{D}}_1$ is the only complete digraph with this property. If $\vec{\mathfrak{G}}$ contains a digon, then every suborthogonal $\vec{\mathfrak{G}}$-decomposition consists of at most one page. Hence, $\vec{\mathfrak{G}}$ has to be a complete digraph itself.

In the sequel, let $\vec{\mathfrak{G}}$ always be a digraph with non-empty arc set and without digons.

In this paper we shall give a Wilson-type result for suborthogonal $\vec{\mathfrak{G}}$-decompositions, i.e. we prove that even such a decomposition of $\vec{\mathfrak{D}}_n$ exists whenever the necessary conditions (1) and (2) hold and n is sufficiently large. Actually, we will show how to alter Wilson's original work to obtain the claimed result.

2. Eventual periodicity of the sets $\mathscr{S}_{\vec{\mathfrak{G}}}$

Let $\vec{\mathfrak{G}}=(W,B)$ be a digraph with k vertices and $b\geqslant 1$ arcs. By $\mathscr{D}_{\vec{\mathfrak{G}}}$ and $\mathscr{S}_{\vec{\mathfrak{G}}}$ we denote the sets of all integers n such that there is a $\vec{\mathfrak{G}}$-decomposition, resp. a suborthogonal $\vec{\mathfrak{G}}$-decomposition of $\vec{\mathfrak{D}}_n$.

Further, we need the notion of a pairwise balanced design (PBD). Let v be a positive integer and K a set of positive integers. A *pairwise balanced design* $PBD(v,K)$ is a pair $(V,\mathscr{F})$, where V is a v-set and $\mathscr{F}$ is a family of subsets of V (called *blocks*), such that every pair of distinct elements of V lies in exactly one block of $\mathscr{F}$ and all block sizes belong to K.

A set K of positive integers is said to be *PBD-closed* if the existence of a $PBD(v,K)$ implies that v lies in K. In [5] Wilson proved that $\mathscr{D}_{\vec{\mathfrak{G}}}$ is PBD-closed for all digraphs $\vec{\mathfrak{G}}$. The crucial observation is that the sets $\mathscr{S}_{\vec{\mathfrak{G}}}$ have the same property.

Theorem 1. *Let $\vec{\mathfrak{G}}$ be a digraph without digons. The set*

$$\mathscr{S}_{\vec{\mathfrak{G}}} = \{n \in \mathbb{Z}: \text{ there exists a suborthogonal } \vec{\mathfrak{G}}\text{-decomposition of } \vec{\mathfrak{D}}_n\}$$

is PBD-closed.

Proof. Let n be an integer in the PBD-closure of $\mathscr{S}_{\vec{\mathfrak{G}}}$. Hence, there is a pairwise balanced design $(V, \mathscr{F})$ of order n and block sizes from $\mathscr{S}_{\vec{\mathfrak{G}}}$. For each block $F \in \mathscr{F}$, let $\mathscr{G}_F$ be a suborthogonal $\vec{\mathfrak{G}}$-decomposition of the complete digraph $\vec{\mathfrak{D}}_{|F|}$ with vertex set F. Let $\mathscr{G}$ be the union of all the families $\mathscr{G}_F$ $(F \in \mathscr{F})$. Obviously, $\mathscr{G}$ is a $\vec{\mathfrak{G}}$-decomposition of $\vec{\mathfrak{D}}_n$.

We claim that $\mathscr{G}$ is suborthogonal, too. Assume, there are two pages whose union contains two or more digons. If these pages belong to different families $\mathscr{G}_F$ and $\mathscr{G}_{F'}$, this implies that the intersection of F and F' contains at least 3 vertices in contradiction to $(V, \mathscr{F})$ being a pairwise balanced design.

On the other hand, the two pages may not belong to the same family $\mathscr{G}_F$, since each of these families is a suborthogonal $\vec{\mathfrak{G}}$-decomposition of a complete digraph. This concludes the proof. $\square$

A set K of positive integers is *eventually periodic* with period π if, for any $s \in K$, we have $n \in K$ for all sufficiently large integers n satisfying $n \equiv s \bmod \pi$. A deep theorem of Wilson [4] states that a PBD-closed set K is eventually periodic with period $\beta(K) = gcd\{n(n-1): n \in K\}$, and hence, with period $\pi = \lambda\beta(K)$ for any positive integer λ.

By Theorem 1, the sets $\mathscr{S}_{\vec{\mathfrak{G}}}$ are eventually periodic. In the following sections, we shall investigate eventual periods of $\mathscr{S}_{\vec{\mathfrak{G}}}$.

3. Edge colorings of complete graphs

To find $\beta(\mathscr{S}_{\vec{\mathfrak{G}}})$ or at least a multiple of this value, we need some results about edge colorings in complete graphs. Let $\mathfrak{K}_k = (W, E)$ be the complete graph on k vertices. An edge coloring of $\mathfrak{K}_k$ with b colors is a mapping $\gamma: E \to \{1, \dots, b\}$.

For a given edge coloring γ of $\mathfrak{K}_k$, we call a 4-cycle $\mathfrak{C}$ in $\mathfrak{K}_k$ *alternating* if its edges $e_1 = \{w_1, w_2\}$, $e_2 = \{w_2, w_3\}$, $e_3 = \{w_3, w_4\}$ and $e_4 = \{w_4, w_1\}$ satisfy $\gamma(e_1) = \gamma(e_3)$ and $\gamma(e_2) = \gamma(e_4)$. We shall prove the following:

Lemma 2. *Let $\mathfrak{G} = (W, F)$ be a spanning subgraph of $\mathfrak{K}_k = (W, E)$ with edges $e_1, \dots, e_b$. There exists an edge coloring γ of $\mathfrak{K}_k$ such that*

$$\gamma(e_i) = i$$

holds for each of the edges $e_1, \dots, e_b$, and such that the vertices of every alternating 4-cycle in $\mathfrak{K}_k$ induce in $\mathfrak{K}_k$ at most one of the edges $e_1, \dots, e_b$.

The proof requires some preliminary claims.

Claim 3. *Let $\mathfrak{B} = (W, F)$ be a tree and z a vertex of $\mathfrak{B}$. There is a bijection $\phi\colon W\backslash\{z\} \to F$ satisfying $w \in \phi(w)$ for each vertex $w \neq z$ in $\mathfrak{B}$.*

Proof. We use induction on $|W| = k$. For $k = 1$ the statement is trivial. Let $k \geqslant 2$. Assume, z is a leaf in $\mathfrak{B}$ and let y denote its neighbor in $\mathfrak{B}$. Deleting z from $\mathfrak{B}$, we again obtain a tree $\mathfrak{B}'$ with $k-1$ vertices. By the induction hypothesis, there is a bijection $\phi'\colon W\backslash\{y,z\} \to F\backslash\{e\}$, where e denotes the edge $\{y,z\}$, such that $w \in \phi'(w)$ holds for all $w \neq y,z$. Define ϕ by

$$\phi(w) = \begin{cases} \phi'(w) & \text{if } w \neq y,z, \\ e & \text{if } w = y. \end{cases}$$

Obviously, ϕ is the claimed bijection.

Now, let z be an arbitrary vertex in $\mathfrak{B}$ and let y be a leaf in $\mathfrak{B}$. Then, there is a bijection $\phi'\colon W\backslash\{y\} \to F$ satisfying $w \in \phi'(w)$ for all $w \neq y$. Let $y = y_0, y_1, \ldots, y_t = z$ be a path from y to z in $\mathfrak{B}$. Define ϕ by

$$\phi(w) = \begin{cases} \phi'(y_{i+1}) & \text{if } w = y_i \ (i = 0, \ldots, t-1), \\ \phi'(w) & \text{if } w \neq y_0, \ldots, y_t. \end{cases}$$

It is easy to see, that ϕ has the claimed property. □

Claim 4. *Let $\mathfrak{G} = (W, F)$ be a connected graph, but not a tree. There is an injection $\phi\colon W \to F$ satisfying $w \in \phi(w)$ for each vertex w in $\mathfrak{G}$.*

Proof. Let $\mathfrak{B}$ be a spanning subtree of $\mathfrak{G}$. Since $\mathfrak{G}$ is not a tree, there is an edge $e = \{y,z\}$ in $\mathfrak{G}$, which is not in $\mathfrak{B}$. By Claim 3, there is an injection $\phi'\colon W\backslash\{z\} \to F\backslash\{e\}$ such that $w \in \phi'(w)$ holds for all vertices $w \neq z$. Define ϕ by

$$\phi(w) = \begin{cases} e & \text{if } w = z, \\ \phi'(w) & \text{otherwise.} \end{cases}$$

Obviously, ϕ is the claimed mapping. □

Now, we are ready to prove Lemma 2.

Proof of Lemma 2. In every tree component of $\mathfrak{G} = (W, F)$ choose a leaf or a singleton, and let Z denote the set of all these vertices. Using Claims 3 and 4, it is easy to verify that there exists an injection $\phi\colon W\backslash Z \to F$ satisfying $w \in \phi(w)$ for all vertices $w \in W$.

Initially, let all edges of $\mathfrak{K}_k$ be uncolored. First, color the edges $e_1, \ldots, e_b$ by $\gamma(e_i) = i$ $(i = 1, \ldots, b)$. Next, for each vertex $w \in W\backslash Z$, we color every uncolored edge e where $w \in e$ by $\gamma(e) = \gamma(\phi(w))$. So far, we have colored all edges with at least one vertex in $W\backslash Z$. Finally, the remaining edges of $\mathfrak{K}_k$ may be colored arbitrarily.

Obviously, whenever two edges of $\mathfrak{K}_k$ have the same color, then they share a vertex in $W\backslash Z$, or at least one of both edges completely lies in Z. Assume, there is an alternating 4-cycle in $\mathfrak{K}_k$ with vertices $w_1, \ldots, w_4$ (Fig. 1). This implies $\{w_1, w_2\} \subseteq Z$

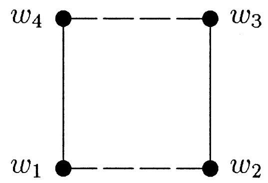

Fig. 1. An alternating 4-cycle.

or $\{w_3, w_4\} \subseteq Z$, as well as $\{w_2, w_3\} \subseteq Z$ or $\{w_4, w_1\} \subseteq Z$. Without loss of generality we may assume, that the vertices w_1, w_2 and w_3 belong to Z. However, every vertex in $W \backslash Z$ has at most one neighbor in Z with respect to $\mathfrak{G}$. Hence, at most one of the pairs $\{w_1, w_4\}, \{w_2, w_4\}, \{w_3, w_4\}$ is an edge in $\mathfrak{G}$. This concludes the proof. $\square$

Note, that whenever the digraph $\mathfrak{G}$ has at most one tree component, the coloring γ constructed in the proof of Lemma 2 has a nice property, namely for any two edges e and e' in $\mathfrak{K}_k$ with $\gamma(e) = \gamma(e')$ we have $e \cap e' \neq \emptyset$. In some sense one may feel that this is an anti-concept to what one usually considers as a good edge coloring. However, it turns out that our edge colorings are helpful for direct constructions of suborthogonal $\vec{\mathfrak{G}}$-decompositions.

4. A direct construction

Using Lemma 2, we are able to prove the following result which is the analogue to Wilson's Proposition 1 in [5].

Lemma 5. *Let $\vec{\mathfrak{G}} = (W, B)$ be a digraph without digons and with $b \geqslant 1$ arcs. Then, a prime power $q \equiv 1 \bmod 2b$ belongs to $\mathscr{S}_{\vec{\mathfrak{G}}}$ whenever q is sufficiently large.*

Proof. Consider $V = \{0, \dots, q-1\}$ to be the finite field $GF(q)$ of order q. Let ω be a generator of the multiplicative group $(V \backslash \{0\}, \cdot)$, and let T be its subgroup of index b and cardinality $(q-1)/b$ generated by ω^b. The cosets of T are $T_1, \dots, T_b$, given by $T_i = \omega^i T$ $(i = 1, \dots, b)$. Note, that the field element -1 lies in T, since $(q-1)/b$ is even.

Let $W = \{1, \dots, k\}$ and $\mathfrak{K}_k$ be the complete graph with vertex set W. Further, let $\mathfrak{G}$ be the undirected underlying graph of $\vec{\mathfrak{G}}$. Obviously, $\mathfrak{G}$ is a spanning subgraph of $\mathfrak{K}_k$. Let γ be the coloring proposed in Lemma 2. Denoting the edges of $\mathfrak{G}$ by $e_1, \dots, e_b$, we have $\gamma(e_i) = i$ for all $i = 1, \dots, b$.

A *k-choice* C is a mapping of $\{(i, j) \in W \times W : i \neq j\}$ into the family of cosets of T. Define the k-choice C by

$$C(i, j) = T_{\gamma(\{i, j\})}, \tag{3}$$

for any two distinct elements i and j from W. Assume, q is sufficiently large. By [3], there is a k-set $X=\{x_1,\dots,x_k\}\subseteq V$ such that $x_j-x_i\in C(i,j)$ holds for all pairs (i,j) with $1\leqslant i<j\leqslant k$. Due to our choice of C, we also have $x_j-x_i\in C(i,j)$ if $1\leqslant j<i\leqslant k$. In particular, the differences x_j-x_i taken over the b arcs (i,j) in $\vec{\mathfrak{G}}$ form a system of representatives of the cosets of T.

Now, for every $v\in V$ and every $t\in T$, we construct a digraph $\vec{\mathfrak{G}}(v,t)$ as the isomorphic image of $\vec{\mathfrak{G}}$ under the mapping $w\to tx_w+v$ for all vertices $w\in W$.

First, we shall show that these digraphs form a $\vec{\mathfrak{G}}$-decomposition $\mathscr{G}$ of $\vec{\mathfrak{D}}_q$. Note, that we have got the right size for the family $\mathscr{G}$. There are just $|V||T|=q(q-1)/b$ pages in $\mathscr{G}$, each of them containing b arcs. On the other hand, the complete digraph $\vec{\mathfrak{D}}_q$ has exactly $q(q-1)$ arcs. Thus, $\mathscr{G}$ is a decomposition of $\vec{\mathfrak{D}}_q$ if every arc of $\vec{\mathfrak{D}}_q$ occurs in at least (hence exactly) one page. For the arc (y,z) in $\vec{\mathfrak{D}}_q$, let (i,j) be the unique arc in $\vec{\mathfrak{G}}$ such that the difference x_j-x_i lies in the same coset of T as the difference $z-y$. Choose $t=(z-y)(x_j-x_i)^{-1}$, which lies in T, and $v=y-tx_i$, which belongs to V. It is easy to check, that (y,z) is contained in the uniquely determined page $\vec{\mathfrak{G}}(t,v)$.

It remains to show, that due to our choice of C, the $\vec{\mathfrak{G}}$-decomposition $\mathscr{G}$ is suborthogonal. Assume, there are two distinct pages $\vec{\mathfrak{G}}(v,t)$ and $\vec{\mathfrak{G}}(v',t')$ whose union contains two or more digons. Hence, there are two distinct arcs (i,j) and (l,m) in $\vec{\mathfrak{G}}$ such that

$$tx_i+v=t'x_j+v',\quad tx_l+v=t'x_m+v',$$

$$tx_j+v=t'x_i+v',\quad tx_m+v=t'x_l+v'.$$

This implies $t=-t'$, and $x_l-x_i=x_j-x_m$.

Hence, $i=l$ holds iff $j=m$ holds, and $i=m$ holds iff $j=l$ holds. Assume, the vertices i, j, l and m are not pairwise distinct. Then, the arcs (i,j) and (l,m) are either equal or form a digon. However, both cases contradict the assumptions.

Hence, $\{i,\ j,\ l,\ m\}$ is a 4-set. In particular, this implies

$$\gamma(\{i,l\})=\gamma(\{j,m\})\quad\text{as well as } \gamma(\{i,m\})=\gamma(\{j,l\}),$$

i.e. the 4-cycle with vertices i, j, l, m in $\mathfrak{K}_k$ is alternating. By Lemma 2, at most one of the pairs $\{i,j\}$ and $\{l,m\}$ is an edge in $\mathfrak{G}$, which gives a contradiction. This concludes the proof. $\square$

With the help of Lemma 5, we are able to determine an eventual period of $\mathscr{S}_{\vec{\mathfrak{G}}}$. Let $\mathscr{Q}_{\vec{\mathfrak{G}}}$ be the set of all prime powers $q\equiv 1 \bmod 2b$ which are large enough in the sense of Lemma 5. By Dirichlet's theorem on primes in arithmetic progressions, $\mathscr{Q}_{\vec{\mathfrak{G}}}$ is non-empty, and so is $\mathscr{S}_{\vec{\mathfrak{G}}}\supseteq\mathscr{Q}_{\vec{\mathfrak{G}}}$. Further, $\beta(\mathscr{Q}_{\vec{\mathfrak{G}}})$ is divisible by $2b$.

Let p be a prime in $\mathscr{Q}_{\vec{\mathfrak{G}}}$ such that $p\equiv 4b+1 \bmod 2b(2b+1)$. This prime exists by Dirichlet's theorem, say $p=2b(2b+1)\xi+4b+1$ for some positive integer ξ. Put $\lambda=(2b+1)\xi+2$, and we obtain $p=2b\lambda+1$. Again by Dirichlet's theorem, there is a prime q such that $q\equiv 1+2b \bmod 2bp\lambda$, say $q=2bp\lambda\mu+2b+1$ for some positive integer μ. Note, that $p\leqslant q$, and thus, q lies in $\mathscr{Q}_{\vec{\mathfrak{G}}}$, too. We have $\gcd\{p(p-1),q(q-1)\}=\gcd\{p(p-1),q-1\}=\gcd\{2bp\lambda,2bp\lambda\mu+2b\}=2b\gcd\{p\lambda,p\lambda\mu+1\}=2b$.

Thus, $2b$ is divisible by $\beta(\mathscr{Q}_{\vec{\mathfrak{G}}}) = \gcd\{n(n-1) : n \in \mathscr{Q}_{\vec{\mathfrak{G}}}\}$. Since $\mathscr{Q}_{\vec{\mathfrak{G}}} \subseteq \mathscr{S}_{\vec{\mathfrak{G}}}$, $\beta(\mathscr{Q}_{\vec{\mathfrak{G}}})$ is divisible by $\beta(\mathscr{S}_{\vec{\mathfrak{G}}})$, which implies

Corollary 6. *$\mathscr{S}_{\vec{\mathfrak{G}}}$ is eventually periodic with period $2b$.*

Note, that using a similar argument, $\mathscr{D}_{\vec{\mathfrak{G}}}$ is eventually periodic with period $2b$, too.

5. A recursive construction

According to Corollary 6, it remains to prove that $\mathscr{S}_{\vec{\mathfrak{G}}}$ contains a representative from each admissible residue class modulo $2b$. The following lemma is the analogue to Wilson's Proposition 3 in [5].

Lemma 7. *Let $\vec{\mathfrak{G}} = (W,B)$ be a digraph without digons and with $b \geqslant 1$ arcs. For every integer s satisfying the necessary conditions* (1) *and* (2), *there is an integer $n \equiv s \bmod 2b$ such that $n \in \mathscr{S}_{\vec{\mathfrak{G}}}$.*

Proof. By Wilson's original claim [5], there is an integer $r \equiv s \bmod 2b$ such that $r \in \mathscr{D}_{\vec{\mathfrak{G}}}$, i.e. such that there exists a $\vec{\mathfrak{G}}$-decomposition $\mathscr{H}$ of the complete digraph $\vec{\mathfrak{D}}_r$. However, $\mathscr{H}$ will usually not be suborthogonal.

Starting with the $\vec{\mathfrak{G}}$-decomposition $\mathscr{H}$ we shall construct a suborthogonal $\vec{\mathfrak{G}}$-decomposition of a larger complete digraph $\vec{\mathfrak{D}}_n$.

First, we present some notation to be used in the sequel. Let $p \geqslant r^2$ be a prime such that $p \equiv 1 \bmod 2b$ and $q = p^\delta \in \mathscr{S}_{\vec{\mathfrak{G}}}$, where $\delta = \binom{r}{2}$. Choose $n = rq$ with $n \equiv r \equiv s \bmod 2b$. Note, that n satisfies the necessary conditions (1) and (2). Our objective is to present a suborthogonal $\vec{\mathfrak{G}}$-decomposition of $\vec{\mathfrak{D}}_n$.

Put $U = \{1,\ldots,r\}$, $Y = \{0,\ldots,p-1\}$, $X = Y^\delta$ and $V = U \times X$. Suppose Y to be the finite field $GF(p)$ of order p, and X to be the δ-dimensional vectorspace over $GF(p)$.

Consider the sequence

$$(1,2),\ldots,(1,r),(2,3),\ldots,(2,r),\ldots,(r-1,r)$$

of all pairs (i,j) from $U \times U$ with $1 \leqslant i < j \leqslant r$. The position (starting with 0) of a fixed pair (i,j) in this sequence is given by the function

$$\rho(i,j) = (r-1)(i-1) - \binom{i}{2} + (j-i) - 1.$$

For any vector $\boldsymbol{x} \in X$, let its coordinates be indexed by the pairs in this sequence, i.e. let $\boldsymbol{x} = (x_{(1,2)},\ldots,x_{(1,r)},x_{(2,3)},\ldots,x_{(r-1,r)})$. Further, denote by T the $(\delta-1)$-dimensional subspace $\{\boldsymbol{x} \in X : x_{(1,2)} + \cdots + x_{(r-1,r)} = 0\}$ of X.

Let ω be a generator of the multiplicative group $(Y \backslash \{0\}, \cdot)$ of $GF(p)$. We define a mapping $\zeta : U \times X \to X$ by $\zeta(u,\boldsymbol{x}) = \boldsymbol{z}$ with coordinates

$$z_{(i,j)} = \begin{cases} x_{(i,j)} & \text{if } u = i \text{ or } u = j, \\ \omega^u x_{(i,j)} & \text{otherwise.} \end{cases}$$

Further, for every page $\vec{\mathfrak{H}}$ in $\mathscr{H}$, we define a mapping $\phi_{\vec{\mathfrak{H}}} : U \times Y \to X$ by $\phi_{\vec{\mathfrak{H}}}(u,y) = \boldsymbol{f}$ with coordinates

$$f_{(i,j)} = \begin{cases} y & \text{if } u = j \text{ and } (i,j) \text{ is an arc in } \vec{\mathfrak{H}}, \\ y + \rho(i,j) & \text{if } u = j \text{ and } (j,i) \text{ is an arc in } \vec{\mathfrak{H}}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we shall show how to obtain the pages of the claimed suborthogonal $\vec{\mathfrak{G}}$-decomposition $\mathscr{G}$. First, on each of the sets $\{u\} \times X$ $(u \in U)$, we construct a suborthogonal $\vec{\mathfrak{G}}$-decomposition $\mathscr{G}_u$ of the complete digraph $\vec{\mathfrak{D}}_q$ with vertex set $\{u\} \times X$. The existence of $\mathscr{G}_u$ is ensured by our assumption $q \in \mathscr{S}_{\vec{\mathfrak{G}}}$.

In addition, for every page $\vec{\mathfrak{H}}$ in $\mathscr{H}$, for every vector $\boldsymbol{x} \in X$, for every vector $\boldsymbol{t} \in T$ and for every integer $y \in Y$, we construct a digraph $\vec{\mathfrak{H}}(\boldsymbol{x}, \boldsymbol{t}, y)$ as the isomorphic image of $\vec{\mathfrak{H}}$ under the mapping $u \to (u, \boldsymbol{u})$ with $\boldsymbol{u} = \boldsymbol{x} + \zeta(u, \boldsymbol{t}) + \phi_{\vec{\mathfrak{H}}}(u, y)$. Denote the family of all these digraphs by $\mathscr{G}_0$.

First, we shall show that the union $\mathscr{G}$ of $\mathscr{G}_0$ and the families $\mathscr{G}_u$ $(u \in U)$ is a $\vec{\mathfrak{G}}$-decomposition of the complete digraph $\vec{\mathfrak{D}}_n$. Note, that we have got the right size for $\mathscr{G}$. Each of the families $\mathscr{G}_u$ contains $q(q-1)/b$ pages, the family $\mathscr{G}_0$ consists of

$$|\mathscr{H}||X||T||Y| = r\frac{r-1}{b}p^{\delta}p^{\delta-1}p = r\frac{r-1}{b}p^{2\delta} = r\frac{r-1}{b}q^2$$

digraphs. The total number of digraphs in $\mathscr{G}$ is

$$rq\frac{q-1}{b} + r\frac{r-1}{b}q^2 = rq\frac{rq-1}{b} = n\frac{n-1}{b},$$

where each of them has exactly b arcs. On the other hand, the complete digraph $\vec{\mathfrak{D}}_n$ contains just $n(n-1)$ arcs.

Thus, we have to verify that every arc of $\vec{\mathfrak{D}}_n$ occurs in at least (and hence exactly) one of the digraphs in $\mathscr{G}$. Consider the arc $((i,\boldsymbol{i}),(j,\boldsymbol{j}))$. If $i = j$, both vertices of the arc lie in the same set $\{u\} \times X$. Hence, the arc belongs to a digraph in $\mathscr{G}_u$, which is a $\vec{\mathfrak{G}}$-decomposition of the complete digraph on $\{u\} \times X$.

If $i < j$, we are looking for a digraph $\vec{\mathfrak{H}}$ from $\mathscr{H}$, vectors $\boldsymbol{x}$, $\boldsymbol{t}$ and an integer y such that $((i,\boldsymbol{i}),(j,\boldsymbol{j}))$ lies in $\vec{\mathfrak{H}}(\boldsymbol{x},\boldsymbol{t},y)$. Let $\vec{\mathfrak{H}}$ be the unique digraph in $\mathscr{H}$ which contains the arc (i,j). Further, let y be the coordinate indexed with (i,j) in the vector $\boldsymbol{j} - \boldsymbol{i}$, i.e. $y = (\boldsymbol{j} - \boldsymbol{i})_{(i,j)}$. Now, we are able to determine the vector $\boldsymbol{z} = \boldsymbol{j} - \boldsymbol{i} - \phi_{\vec{\mathfrak{H}}}(j,y) + \phi_{\vec{\mathfrak{H}}}(i,y)$. Choose for any coordinate indexed with $(u,v) \neq (i,j)$ in $\boldsymbol{t}$

$$t_{(u,v)} = \begin{cases} (\omega^j - 1)^{-1} z_{(u,v)} & \text{if } u = i \text{ or } v = i, \\ (1 - \omega^i)^{-1} z_{(u,v)} & \text{if } u = j \text{ or } v = j, \\ (\omega^j - \omega^i)^{-1} z_{(u,v)} & \text{otherwise.} \end{cases}$$

The remaining coordinate $t_{(i,j)}$ of $\boldsymbol{t}$ should be chosen such that $\boldsymbol{t}$ lies in T, i.e. such that $t_{(1,2)} + \cdots + t_{(i,j)} + \cdots + t_{(r-1,r)} = 0$ holds in $GF(p)$. Thus, we obtain $\boldsymbol{z} = \zeta(j,\boldsymbol{t}) - \zeta(i,\boldsymbol{t})$. Finally, let $\boldsymbol{x} = \boldsymbol{i} - \zeta(i,\boldsymbol{t}) - \phi_{\vec{\mathfrak{H}}}(i,y)$.

Similarly, we handle the case $j < i$. It is not difficult to check, that the arc $((i,\boldsymbol{i}),(j,\boldsymbol{j}))$ is contained in the uniquely determined digraph $\vec{\mathfrak{H}}(\boldsymbol{x},\boldsymbol{t},y)$.

It remains to verify the suborthogonality of $\mathscr{G}$. Assume, there are two pages in $\mathscr{G}$ whose union contains two or more digons. Obviously, both pages should belong to the family $\mathscr{G}_0$. Hence, denote them by $\vec{\mathfrak{H}}(\boldsymbol{x},\boldsymbol{t},y)$ and $\vec{\mathfrak{H}}'(\boldsymbol{x}',\boldsymbol{t}',y')$. Since their union contains two digons, there are two different arcs $((i,\boldsymbol{i}),(j,\boldsymbol{j}))$ and $((l,\boldsymbol{l}),(m,\boldsymbol{m}))$ in $\vec{\mathfrak{H}}(\boldsymbol{x},\boldsymbol{t},y)$ such that the reverse arcs $((j,\boldsymbol{j}),(i,\boldsymbol{i}))$ and $((m,\boldsymbol{m}),(l,\boldsymbol{l}))$ lie in $\vec{\mathfrak{H}}'(\boldsymbol{x}',\boldsymbol{t}',y')$. In particular, this implies that the digraph $\vec{\mathfrak{H}}$ contains arcs (i,j) and (l,m) as well as the digraph $\vec{\mathfrak{H}}'$ contains arcs (j,i) and (m,l).

Due to the construction of the pages in $\mathscr{G}_0$, we have

$$\begin{aligned}
\boldsymbol{i} &= \boldsymbol{x} + \zeta(i,\boldsymbol{t}) + \phi_{\vec{\mathfrak{H}}}(i,y) = \boldsymbol{x}' + \zeta(i,\boldsymbol{t}') + \phi_{\vec{\mathfrak{H}}'}(i,y'),\\
\boldsymbol{j} &= \boldsymbol{x} + \zeta(j,\boldsymbol{t}) + \phi_{\vec{\mathfrak{H}}}(j,y) = \boldsymbol{x}' + \zeta(j,\boldsymbol{t}') + \phi_{\vec{\mathfrak{H}}'}(j,y'),\\
\boldsymbol{l} &= \boldsymbol{x} + \zeta(l,\boldsymbol{t}) + \phi_{\vec{\mathfrak{H}}}(l,y) = \boldsymbol{x}' + \zeta(l,\boldsymbol{t}') + \phi_{\vec{\mathfrak{H}}'}(l,y'),\\
\boldsymbol{m} &= \boldsymbol{x} + \zeta(m,\boldsymbol{t}) + \phi_{\vec{\mathfrak{H}}}(m,y) = \boldsymbol{x}' + \zeta(m,\boldsymbol{t}') + \phi_{\vec{\mathfrak{H}}'}(m,y').
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\boldsymbol{j} - \boldsymbol{i} &= \zeta(j,\boldsymbol{t}) - \zeta(i,\boldsymbol{t}) + \phi_{\vec{\mathfrak{H}}}(j,y) - \phi_{\vec{\mathfrak{H}}}(i,y)\\
&= \zeta(j,\boldsymbol{t}') - \zeta(i,\boldsymbol{t}') + \phi_{\vec{\mathfrak{H}}'}(j,y') - \phi_{\vec{\mathfrak{H}}'}(i,y').
\end{aligned}$$

Without loss of generality, let $i < j$. Considering the coordinate of $\boldsymbol{j} - \boldsymbol{i}$ indexed with (i,j), this implies

$$\begin{aligned}
& t_{(i,j)} - t_{(i,j)} + y - 0 = t'_{(i,j)} - t'_{(i,j)} + (y' + \rho(i,j)) - 0,\\
& y = y' + \rho(i,j). \qquad (4)
\end{aligned}$$

Similarly, considering the coordinate indexed with (l,m) (or (m,l), respectively), of the vector $\boldsymbol{m} - \boldsymbol{l}$, we obtain

$$y = \begin{cases} y' + \rho(l,m) & \text{if } l < m,\\ y' - \rho(m,l) & \text{if } m < l. \end{cases}$$

Together with (4), this implies

$$\rho(i,j) = \begin{cases} \rho(l,m) & \text{if } l < m,\\ -\rho(m,l) & \text{if } m < l \end{cases} \qquad (5)$$

in the finite field $GF(p)$.

However, by the definition of the function ρ, we have

$$0 \leqslant \rho(u,v) < \binom{r}{2} < \frac{p}{2},$$

for every pair (u,v) of integers with $1 \leqslant u < v \leqslant r$. Therefore, Eq. (5) implies

$$(i,j) = \begin{cases} (l,m) & \text{if } l < m,\\ (m,l) & \text{if } m < l, \end{cases}$$

i.e. both cases give a contradiction to our assumption. This concludes the proof. $\square$

As an immediate consequence of Lemma 7 we obtain our main theorem on the asymptotic existence of suborthogonal $\vec{\mathfrak{G}}$-decompositions.

Theorem 8. *Let* $\vec{\mathfrak{G}}$ *be a digraph without digons and with non-empty arc set. There exists a suborthogonal* $\vec{\mathfrak{G}}$*-decomposition of the complete digraph* $\vec{\mathfrak{D}}_n$ *for all sufficiently large integers n satisfying the necessary conditions* (1) *and* (2).

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